Invariant cones for linear elliptic systems with gradient couplings

Introduction

We consider **smooth** vector-valued functions $u = (u_1, \dots, u_m)$ of the variable x in a bounded open subset $\Omega \subset \mathbb{R}^n$ satisfying linear systems of partial differential inequalities with constant coefficients of the following form

$$\begin{cases} \mathrm{Tr}(A\nabla^2 u_1) + \sum_{k=1}^m b_1^k \cdot \nabla u_k + \sum_{k=1}^m c_{1k} u_k \geq 0 \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ \mathrm{Tr}(A\nabla^2 u_m) + \sum_{k=1}^m b_m^k \cdot \nabla u_k + \sum_{k=1}^m c_{mk} u_k \geq 0 \end{cases} \tag{1}$$

or, in compact form,

$$Au + \sum_{i=1}^n B^{(i)} D_i u + Cu \geq 0 \text{ in } \Omega \tag{2}$$

Introduction

Here A is the second order matrix operator

$$Au = \begin{pmatrix} \mathrm{Tr}(A\nabla^2 u_1) \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Tr}(A\nabla^2 u_m) \end{pmatrix} \tag{3}$$

where A is a positive semidefinite $n \times n$ matrix, $B^{(i)}$ and C are $m \times m$, which I assume for simplicity to have constant entries, and for $i = 1, \ldots, n$,

$$D_i u = \begin{pmatrix} \frac{\partial u_1}{\partial x_i} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial u_m}{\partial x_i} \end{pmatrix} \tag{4}$$

denotes the $i - th$ column of the Jacobian matrix of the vector function u.

Note that the above defined structure of the system **allows coupling between the u_j and their gradients but not at the level of second derivatives.**

Specific assumptions on A, $B^{(i)}$ and C will be made later on.
We will often refer to a solution of (17) as a **subsolution** of the system

Systems of this kind naturally arise in several different contexts such as modeling of simultaneous diffusions of m substances which decay spontaneously or in the case of systems describing switching diffusion processes in probability theory.
In the latter case the homogeneous Dirichlet problem for system (17) describes discounted exit times from Ω.

Introduction

Question: does the **weak Maximum Principle** (**wMP**) holds for such systems ?

that is find conditions on the data such that the **sign propagation property from the boundary to the interior**:

$$u_i \leq 0 \text{ on } \partial\Omega \text{ for all } i = 1 \dots m \quad \text{implies} \quad u_i \leq 0 \text{ in } \Omega \text{ for all } i = 1 \dots, m \tag{5}$$

holds true in a generic bounded domain $\Omega \subset \mathbb{R}^n$?

The scalar case: (wMP) and the sign of the principal eigenvalue

For general **linear uniformly elliptic** operators in **non variational form** with continuous coefficients

$$Lu = \mathrm{Tr}\big(A(x)\nabla^2 u\big) + b(x)\cdot\nabla u + c(x)u$$

Berestycki-Nirenberg-Varadhan CPAM 1994 considered the number

$$\lambda_1(L;\Omega) := \sup[\lambda : \exists\varphi > 0 \text{ such that } (L+\lambda)\varphi \le 0] = -\inf_{\varphi>0}\sup_{\Omega}\frac{L\varphi}{\varphi} \qquad (6)$$

(see also in this respect Donsker-Varadhan PNAS 1975, Venturino BUMI 1978 and the Collatz-Wielandt formula in finite dimensional spaces, see Beckenbach-Bellman Springer 1961).

They showed that:

- $\lambda_1(L;\Omega)$ is **the principal eigenvalue** for the homogeneous Dirichlet problem for L in Ω (i.e. λ_1 is real and simple with an associated positive eigenfunction and for any other eigenvalue $Re\lambda \ge \lambda_1$)
- **(wMP)** holds for $W^{2,p}$ functions **if and only if** $\lambda_1(L;\Omega) > 0$

The scalar case: (wMP) and the sign of the principal eigenvalue

The Berestycki-Nirenberg-Varadhan result has been later extended to different situations such as:

1. A **positive semidefinite** and, more generally, **fully nonlinear degenerate elliptic** operators
   (note that in this case $\lambda_1(L;\Omega)$ as defined in (6) may not be a genuine eigenvalue), Berestycki-CD-Porretta-Rossi JMPA 2015
2. fully nonlinear uniformly elliptic operators, Birindelli-Demengel Adv.Differ.Equ. 2006

The scalar case: (wMP) and the sign of the principal eigenvalue

As we will see later one item 1 will be relevant with respect to the theme of this talk:

one consequence of the main result in Berestycki-CD-Porretta-Rossi 2015 is that **(wMP)** holds for **viscosity solutions** $v \in C(\Omega)$ of the **nonlinear scalar** inequality

$$max_{k\in\{1,\dots m\}} L_k v \geq 0 \text{ in } \Omega \quad , \quad v \leq 0 \text{ on } \partial\Omega \tag{7}$$

provided that the numerical index evaluated on smooth **positive** functions φ by the formula

$$\mu_1 := -\inf_{\varphi>0} \sup_{\Omega} \frac{\max_{k\in\{1,\dots m\}} L_k\varphi(x)}{\varphi(x)}$$

is **strictly positive**.

The vector case, no couplings in first order terms

Consider here the **weakly coupled** system

$$\begin{cases} \mathrm{Tr}(A_1 \nabla^2 u_1) + b_1 \cdot \nabla u_1 + \sum_{k=1}^{m} c_{1k} u_k \geq 0 \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{Tr}(A_m \nabla^2 u_m) + b_m \cdot \nabla u_m + \sum_{k=1}^{m} c_{mk} u_k \geq 0 \end{cases} \tag{8}$$

Observe that the system is **weakly coupled**, meaning that coupling occurs only in the zero-order terms.
In the vector case the (**wMP**) does not hold true in general: a simple counterexample is :

let $m = 2$ and $A_1 = A_2 = I$ and $b_1 = b_2 \equiv 0$, so that the second order operators both reduce to Laplace and consider

$$C = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

The vector case, no couplings in first order terms

The functions

$$u_1 = 1 - |x|^2 \quad , \quad u_2 = -2n$$

solve the associated system

$$\Delta u_1 - u_2 = 0 \quad , \quad \Delta u_2 = 0$$

in the open unit ball $B_1 \subset \mathbb{R}^n$ with $u_1 = 0$, $u_2 < 0$ on ∂B_1 but

$$u_1 > 0 \quad in \quad B_1$$

As it expected from what well-known in the scalar case, the validity of **(wMP)** **requires conditions on the matrix** C.

The vector case: weak couplings and cooperative matrices

As we will see soon after, an appropriate condition which guarantees the validity of (**wMP**) in the weakly coupled setting is that of **cooperative** matrix, meaning that

$$c_{ik} \geq 0 \quad \forall\, i \neq k, \quad \sum_{k=1}^{m} c_{ik} \leq 0, \quad i = 1 \dots m\,, \tag{9}$$

Observe that this implies that

$$c_{ii} \leq 0 \quad \text{and the dominance relation} \quad c_{ii} \leq -\sum_{k \neq i} c_{ik}$$

for all $i = 1 \dots m$.
So, in the trivial **uncoupled case** the familiar conditions $c_{ii} \leq 0$ for the validity of (**wMP**) in the scalar case are recovered.

Observe also that cooperativity is **not satisfied** in the previous counterexample since $c_{12} < 0$.

The vector case: weak couplings and cooperative matrices

As the name suggests the notion of cooperativity originates in Game Theory in the framework of stationary or discrete time models involving a finite number of agents.

In the framework of differential models it has been considered by various authors, see e.g. Hirsch (1985).
It is worth to observe that **cooperativity** implies that the initial conditions $y_i(0) \le 0$ continue to hold at later times, namely $y_i(t) \le 0$ under the flow $dy/dt = Cy\,,\, t > 0$.

Concerning pde's, the condition of cooperativity is used (with no reference to that name) in the 1967 book by Protter-Weinberger to prove the validity of the **(wMP)** for **weakly coupled uniformly parabolic or elliptic** systems such as (8).

The vector case: weak couplings and cooperative matrices

Several different generalizations and developments of this result, always under **uniform ellipticity** of the principal part of the linear operator, were established in later papers such as:

- G. Sweers (1992)
- I. Birindelli, E. Mitidieri, G. Sweers (1999)
- J. Busca, B. Sirakov (2004)
- H. Amann (2005)

In a recent paper with A. Vitolo (Journal of Convex Analysis Volume 28 (2021) dedicated to Umberto Mosco) we obtained **(wMP)** results for **weakly coupled cooperative degenerate elliptic** systems, both linear or fully nonlinear.
Our method of proof is different from those adopted in the above mentioned papers since it is based on a reduction to the scalar case:
for $k = 1, \ldots, m$ let

$$L_k\varphi = \mathrm{Tr}(A\nabla^2\varphi) + b_k \cdot \nabla\varphi$$

and consider the **nonlinear, scalar** Bellman type operator

$$F^*[\varphi] = \max_{k\in 1,\ldots,m} L_k\varphi$$

This construction has been considered in J. Busca, B. Sirakov (2004).

Weak couplings; reduction to the scalar case

In CD-Vitolo (2021) we proved that **(wMP)** holds for system (8) provided matrices A and C are, respectively, **positive semidefinite** and **cooperative**.

The main steps are:

- if C is **cooperative** and $u = (u_1, \ldots, u_m)$ is a (smooth/Sobolev) solution of system (8) with $u_k \leq 0$ on $\partial\Omega$, then the **scalar function**

$$u^* := \max_{k=1,\ldots,m} u_k^+$$

  satisfies in the **viscosity sense** the **nonlinear scalar** inequality

$$F^*[u^*] \geq 0 \text{ in } \Omega \qquad \text{and} \quad u^* = 0 \text{ on } \partial\Omega \tag{10}$$

  where F^* is the **degenerate elliptic convex** operator given by

$$F^* = \max_{k=1,\ldots,m} L_k$$

  Here is where **cooperativity** plays a role. This step requires of course some elementary **viscosity solutions** tools.
  More details in this respect in the final slides.

Weak couplings; reduction to the scalar case

Next steps:

- if A_k are positive semidefinite with $\mathrm{Tr} A_k > 0$, then there exist $\overline{\varphi} > 0$ such that
$$L_k\overline{\varphi} + \lambda\overline{\varphi} \leq 0 \quad \text{for sufficiently small } \lambda > 0$$

- the above implies that the numerical index
$$\mu_1(F^*) := -\inf_{\varphi>0}\sup_{x\in\Omega}\frac{\max_{k\in\{1,\dots m\}} L_k\varphi(x)}{\varphi(x)}$$
is **strictly positive**

- an equivalent expression for $\mu_1(F^*)$ is
$$\mu_1(F^*) = \sup[\lambda : \exists\varphi > 0 \text{ such that } (F^* + \lambda)\varphi \leq 0]$$

- by Berestycki,CD,Porretta,Rossi (2015) this implies validity of (**wMP**) for (10) and it easily follows (recall the definition of u^* !) that (**wMP**) holds for system (8)

The vector case: non diagonal structure in first derivatives

When **coupling in first order terms occurs** in the system simple examples show that the **(wMP)** may indeed fail:

Example

The vector $(u, v) = (1 - x_1^2 - x_2^2, \frac{1}{3}x_1^3 + 4x_2 - 20)$ *is a solution of*

$$\begin{cases} \Delta u + \dfrac{\partial v}{\partial x_2} = 0 \\ \Delta v + \dfrac{\partial u}{\partial x_1} = 0 \end{cases}$$

in the unit ball $\Omega \subset \mathbb{R}^2$, $u = 0$, $v < 0$ *on* $\partial\Omega$ *but* $u > 0$ *in* Ω.

Observe that the zero-order matrix is $C \equiv 0$ *in this example, so that cooperativity is fulfilled.*

Non diagonal structure in first derivatives: counterexamples

As a matter of fact, even a **first-order coupling of arbitrarily small size** in the system can be responsible of the loss of (**wMP**), as the following example shows:

Example

The system

$$\begin{cases} \Delta u + \varepsilon \dfrac{\partial v}{\partial x_1} \geq 0 \\ \Delta v + \varepsilon' \dfrac{\partial u}{\partial x_1} \geq 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, *fulfills* (**wMP**) *if and only if* $\varepsilon = \varepsilon' = 0$. *Indeed the validity of* (**wMP**) *when* $\varepsilon = \varepsilon' = 0$ *is classical. Conversely, if, say,* $\varepsilon \neq 0$, *then* (**wMP**) *is violated by the pair*

$$u(x) = \delta - |x - \bar{x}|^2, \qquad v(x) = v(x_1) = e^{\sigma x_1} - H,$$

by choosing $\bar{x} \in \Omega$ *and* $\delta > 0$ *small enough to have* $u < 0$ *on* $\partial\Omega$, *then take* σ *with the same sign as* ε *and with* $|\sigma|$ *large enough to have the two inequalities in the system, and finally* H *large enough to have* $v < 0$ *on* $\partial\Omega$.

Remark.
The above example enlightens an instability property of **(wMP)** *for cooperative systems with respect to first order perturbations.*
This is in striking contrast with the scalar case: indeed, for a uniformly elliptic scalar inequality, not only the presence of a first order term does not affect the validity of **(wMP)** *when the zero-order term is nonpositive, but in addition* **(wMP)** *is* **stable with respect to perturbations of the coefficients**, *in the* L^{∞} *norm.*

This can be seen as a consequence of the fact that in the scalar case **(wMP)** *is characterized by the positivity of the associated principal eigenvalue, and the latter depends continuously on the coefficients of the operator, see Berestycki-Nirenberg-Varadhan (1994).*

In the case of weakly coupled systems their notion of principal eigenvalue was extended by Birindelli-Mitidieri-Sweers (1999) to cooperative systems **without first-order couplings**. *The above example reveals then that either such a notion does not exist when there is a first-order coupling, or that it is not continuous with respect to the interaction coefficients.*

Non diagonal structure in first derivatives: counterexamples

The next, more involved, example with $m = 2$ and $n = 1$ shows that (**wMP**) fails when the **diagonal zero-order term c is sufficiently large or when the size ρ of the interval is sufficiently small**.
Observe also that the example shows that enforcing a **"stronger" form of cooperativity is not enough** to guarantee (**wMP**).

This can be surprising if one has in mind the picture for the scalar equation (where a large negative zero-order term and/or a small domain enforce the validity of the maximum principle).

Example

Let $\varepsilon > 0$, $\alpha, \tilde{c} \geq 0$. Then the system

$$\begin{cases} u'' \pm \varepsilon v' - cu + \alpha v \geq 0 \\ v'' - \tilde{c} v \geq 0 \qquad\qquad x \in I_\rho = (0, \rho) \end{cases} \tag{11}$$

where u,v are functions of the single variable $x \in \mathbb{R}$ does not satisfy (**wMP**), *provided that*

$$\zeta(\rho\sqrt{c})\sqrt{c} > \frac{\alpha}{\varepsilon} \quad \text{where} \quad \zeta(\tau) := \frac{\cosh\tau - 1}{\sinh\tau - \tau}\,. \tag{12}$$

Invariant sets in the sense of Weinberger

The previous examples show that there is no hope for general conditions guaranteeing **(wMP)** for cooperative systems when **coupling occur in gradient terms**.

The **(wMP)** property can be understood in the framework of the general theory of **invariant sets** introduced by H. F. Weinberger, Rendiconti di Matematica (1975) (a special issue dedicated to Mauro Picone 90th birthday) in the context of elliptic and parabolic weakly coupled systems.

A recent reference in this direction is G. Kresin-V. Mazya, J.Anal.Math. (2018), where the notion of invariance is thoroughly developed for general systems with **couplings at the first and the second order in the case** $C \equiv 0$.

According to the notion introduced by Weinberger, a set $S \subseteq \mathbb{R}^m$ is **invariant** for the system

$$Au + \sum_{i=1}^{n} B^{(i)} D_i u + Cu \geq 0 \text{ in } \Omega$$

if the following property holds:

$$\textbf{INV} \qquad u(x) \in S \text{ for all } x \in \partial\Omega \implies u(x) \in S \text{ for all } x \in \Omega \qquad (13)$$

Invariant sets in the sense of Weinberger

The sign propagation property **(wMP)** can therefore be rephrased as the property of the **negative orthant**

$$\mathbb{R}^m_- = \{u = (u_1, \ldots u_m) : u_j \leq 0 \ , \ j = 1, \ldots, m\}$$

being an **invariant set** for our system of partial differential inequalities.

Weinberger proved in particular that $\mathbb{R}^m_-$ is **invariant** for **weakly coupled uniformly elliptic** systems such as

$$\mathrm{Tr}(A_j \nabla^2 u_j) + b_j \cdot \nabla u_j + f(u) = 0 \ , \ j = 1, \ldots, m \tag{14}$$

under the condition that the vector field f satisfies the following property:

for any p belonging to the **outward normal cone** to $\mathbb{R}^m_-$ at a point u on the boundary of $\mathbb{R}^m_-$, the inequality

$$p \cdot f(u) \leq 0 \tag{15}$$

holds.

For $f(u) = Cu$, this geometric condition involving the **zero-order terms** and the **geometry of the boundary** of negative orthant turns out to be the **cooperativity property** (9) of matrix C.

Invariant sets in the sense of Weinberger

As seen before $\mathbb{R}^m_-$ may fail to be an invariant set even when the coupling of the first order terms is very small.
Observe also that the matrix of the first order terms in system

$$\begin{cases} u'' \pm \varepsilon v' - cu + \alpha v \geq 0 \\ v'' - \tilde{c}v \geq 0 \qquad\qquad x \in I_\rho = (0, \rho) \end{cases} \tag{16}$$

is

$$\begin{pmatrix} 0 & \pm\varepsilon \\ 0 & 0 \end{pmatrix}$$

which for $\varepsilon > 0$ is **not diagonal nor diagonalizable**.

This is indeed consistent with results in Kresin-Mazya (2018), where the case $C \equiv 0$ is considered. Their conditions involving the geometry of a closed convex set S and the matrices $B^{(i)}$ which imply the invariance of S, necessarily require, in the case $S = \mathbb{R}^m_-$, the **diagonal structure of the first order couplings**.

Invariant sets in the sense of Weinberger: the non-diagonal case

On the account of the previous examples and considerations there is no hope to find general conditions under which **(wMP)** holds when first-order coupling occur.
We are necessarily led to look for appropriate invariant sets, different from $\mathbb{R}^m_-$, for system

$$Au + \sum_{i=1}^{n} B^{(i)} D_i u + Cu \geq 0 \text{ in } \Omega \tag{17}$$

when the $B^{(i)}$ **do not have a diagonal structure**.

A first result in this direction is the following one taken from CD-LR-AV (2021, to appear in PAFA Special Issue dedicated to Louis Nirenberg):
the result shows that under some algebraic conditions, including notably the **simultaneous diagonalizability** of the matrices $B^{(i)}$ and, of course, **cooperativity** one can find **invariant cones** for the above system.

Our approach, which provides as far as we know a first result in this direction, **does not** allow to treat **non constant coefficients** in the matrices A and B^i.

Invariant sets in the sense of Weinberger: the non-diagonal case

Theorem

Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that there exists an invertible $m \times m$ matrix Q such that, for all $i = 1, \ldots, n$,

$$Q^{-1}B^{(i)}Q = \mathrm{Diag}\big(\beta_1^{(i)}, \ldots, \beta_m^{(i)}\big) \quad \text{for some } \beta_j^{(i)} \in \mathbb{R}, \quad (j = 1, \ldots, m) \tag{18}$$

$$Q^{-1} \geq 0 \tag{19}$$

and, moreover,

$$Q^{-1}CQ \ \text{ is cooperative} \tag{20}$$

Assume also that

$$A \text{ is positive semidefinite and } A\nu \cdot \nu \geq \lambda > 0 \text{ for some direction } \nu \in \mathbb{R}^n \tag{21}$$

Then the convex cone $S = \{u \in \mathbb{R}^m : Q^{-1}u \leq 0\}$ is invariant for system (17).

Remark.

One cannot expect, in general, the invariance of the negative orthant R^m_-

If no coupling occurs in first derivatives, so that $Q = Q^{-1} = I$*, then* R^m_- *is indeed invariant and the above result reproduces the one in CD-Vitolo (2020). This is indeed coherent with above mentioned results in Kresin-Mazya (2018).*

Invariant sets in the sense of Weinberger: the non-diagonal case

Remark.
Concerning the linear algebraic conditions of Theorem 1, observe first that a matrix Q *satisfying (18) may be found indeed if the* $B^{(i)}$*'s* **commute** *each other for all* $i = 1, \ldots, n$.

Observe also that if Q *is an* **invertible M-matrix**, *that is* $Q = sI - X$ *where* $X \geq 0$ *and* s *is strictly greater than the spectral radius of* X, *then* Q *fulfills condition (19), see Berman-Plemmons Nonnegative Matrices in the Mathematical Sciences (1994).*

Also, conditions (19) and (20) are compatible.
For example, $Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ *is an invertible M-matrix,*
$C = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$ *is cooperative and* $Q^{-1}CQ = \begin{pmatrix} -4 & 3 \\ 0 & -1 \end{pmatrix}$ *is cooperative as well.*

Remark.
Permutation matrices satisfies both $Q^{-1} \geq 0$ *and* $Q \geq 0$, *so that in this case the conclusion of Theorem 1 is in fact that the negative orthant* R^m_- *is invariant. It is easy to check that in this situation condition (18) implies that each* $B^{(i)}$ *is diagonal and the results of CD-V (2020) apply.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

a very simple permutation matrix

The next toy example provides a simple illustration of the result of Theorem 1:

Example

Let $u = (u_1, u_2)$ *be a solution of*

$$\begin{cases} \Delta u_1 + 6\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_1} - u_1 \geq 0 \\ \Delta u_2 - 8\dfrac{\partial u_1}{\partial x_1} - u_2 \geq 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$*. In this case* $B^{(1)} = \begin{pmatrix} 6 & 1 \\ -8 & 0 \end{pmatrix}$, $B^{(2)} = 0$, $C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *and Theorem 1 applies with*

$$Q = \begin{pmatrix} -1 & 1/2 \\ 4 & -1 \end{pmatrix} \quad Q^{-1} = \begin{pmatrix} 1 & 1/2 \\ 4 & 1 \end{pmatrix}$$

So, inequality $u_2 \leq \min(-2u_1; -4u_1)$ *propagates from* $\partial\Omega$ *to the whole* Ω*.*

To conclude, let us sketch the proof of Theorem 1.

Assume that $u \in [C^2(\Omega)]^m \cap [C^0(\overline{\Omega})]^m$ satisfies system (17) and that vector u is ≤ 0 on $\partial\Omega$. Set

$$\hat{B}^{(i)} := Q^{-1}B^{(i)}Q \quad \text{and} \quad \hat{C} := Q^{-1}CQ.$$

Change the unknown by setting $u = Qv$. This gives, on the account of assumptions (18), (19), that v satisfies

$$Av + \sum_{i=1}^{n} \hat{B}^{(i)} D_i v + \hat{C}v \geq 0 \text{ in } \Omega \text{ and } v \leq 0 \text{ on } \partial\Omega \tag{22}$$

that is, componentwise,

$$\begin{cases} Tr(A\nabla^2 v_1) + b_1 \cdot \nabla v_1 + \hat{C}_1 v \geq 0 \\ \quad \cdots \\ Tr(A\nabla^2 v_m) + b_m \cdot \nabla v_m + \hat{C}_m v \geq 0 \end{cases} \tag{23}$$

where $b^j = (\beta_j^{(1)}, \ldots, \beta_j^{(m)})$ and $\hat{C}_j$ is the j-th row of $\hat{C}$, for $j = 1, \ldots, m$.

We now employ some arguments from CD-Vitolo (2020):

- reduce the above system to a scalar inequality governed by the uniformly elliptic (nonlinear) Bellman operator F^* in (7).
- since the vector $v = (v_1, \ldots, v_m)$ is a **classical solution** of (22) then by viscosity calculus and the cooperativity condition the **scalar function**

$$v^*(x) := \max_{j=1,\ldots,m} (v_j)^+(x) \,,$$

is a **continuous viscosity solution** of the **nonlinear scalar equation**

$$F^*[v^*] \geq 0 \text{ in } \Omega \text{ and } v^* = 0 \text{ on } \partial\Omega \,. \tag{24}$$

Invariant sets in the sense of Weinberger: the non-diagonal case

To check this fact, let a smooth function φ **touch from above** v^* at some point in Ω. If at that point $v^* = 0$ then clearly $F^*[\varphi] \geq 0$ there.
Otherwise, φ touches from above the component v_j realizing the positive maximum v^* at that point aso that

$$Tr(A\nabla^2\varphi) + b_j \cdot \nabla\varphi + \hat{C}_j v \geq 0.$$

Since $\hat{C}$ is cooperative then

$$\hat{C}_j v \leq v_j \sum_k \hat{C}_{jk} \leq 0,$$

whence again $F[\varphi] \geq 0$.

The next step is to apply to equation

$$F^*[v^*] \geq 0 \ \text{ in } \ \Omega \ \text{ and } \ v^* = 0 \text{ on } \ \partial\Omega$$

the general result of BCDPR (2015) .

At this purpose we need to show that the **(pseudo) principal eigenvalue** $\mu_1(F^*, \Omega)$ is **strictly** positive.
This amounts to **finding a strict supersolution which is strictly positive** in $\overline{\Omega}$.
The latter is simply provided by

$$\psi(x) = \psi(x_1, \ldots, x_m) = 1 - \delta e^{\gamma x_1}$$

Indeed,

$$F^*[\psi] = -\delta\gamma e^{\gamma x_1}\left(\gamma + \min_{j=1,\ldots,m} \beta_j^{(1)}\right),$$

which is **strictly negative** in Ω provided $\gamma > |\min_{j=1,\ldots,m} \beta_j^{(1)}|$.
We then choose δ small enough, depending on γ and Ω, so that $\psi > 0$ in $\overline{\Omega}$.

Invariant sets in the sense of Weinberger: the non-diagonal case

Therefore, ψ is positive in $\overline{\Omega}$ and satisfies there $F^*[\psi] < 0$, hence also $F^*[\psi] + \lambda\psi < 0$ for $\lambda > 0$ suitably small.

This implies that the numerical index $\mu_1(F^*, \Omega)$ defined by

$$\mu_1(F^*, \Omega) = \sup\{\lambda \in \mathbb{R} : \psi \in C(\overline{\Omega}),\ \exists\psi > 0,\ F^*[\psi] + \lambda\psi \leq 0 \text{ in } \Omega\} \quad (25)$$

is **strictly positive**.
Hence, according to BCDPR (2015), **(wMP)** holds true for the scalar problem

$$F^*[v^*] \geq 0 \ \text{ in } \ \Omega \ \text{ and } \ v^* = 0 \text{ on } \ \partial\Omega$$

so that

$$v^*(x) \leq 0\,, x \in \Omega$$

It follows easily that $v_j^+ \leq 0$ in Ω.

Recall that by assumption $Q^{-1} \geq 0$ to conclude that

$$Q^{-1}u = v \leq 0 \text{ in } \Omega$$

and the proof is complete.

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